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Some Topics on Frobenius–Lusztig Kernels, II

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We discuss a method to classify all Hopf subalgebras of a certain class of pointed Hopf algebras including quantized universal enveloping algebras and Frobenius–Lusztig kernels. We find factor Hopf algebras of the dual quotients of the coordinate rings of quantum $GL(n)$. We consider a large class of quasitriangular structures for the Frobenius–Lusztig kernels and compute ribbon elements.

Key Words: Frobenius–Lusztig kernels; half quantum groups; sub-Hopf algebras; quasitriangular structures; ribbon elements.

We continue the investigation of restricted specializations and Frobenius–Lusztig kernels from [10].

In Section 6, a new method is described to classify all Hopf subalgebras of a certain class of pointed Hopf algebras. This can be applied to $\mathfrak{u}$ and **U** and gives the result that all Hopf subalgebras of $\mathfrak{u}$ and of **U** are generated as algebras by certain subsets of the canonical algebra generators of $\mathfrak{u}$ and **U**, respectively.

As an application, in Section 7 we describe all factor bialgebras of the Frobenius kernels of $\mathcal{O}_q(SL_n)$ and $\mathcal{O}_q(GL_n)$, the coordinate rings of quantum SL_n and quantum GL_n, respectively, and find their central group-like elements.

In Section 8 we define analogues of the "Quasi-$\mathcal{R}$-Matrix" from [6] for the Frobenius–Lusztig kernels and use it to find a class of quasitriangular structures which contains all quasitriangular structures from the classifica-

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tion in [4]. We show that each quasitriangular structure admits a ribbon element.

6. HOPF SUBALGEBRAS OF FROBENIUS–LUSZTIG KERNELS

We show that all Hopf subalgebras of $\mathbf{U}$ and $\tilde{\mathfrak{u}}$ (under some assumptions on the order of q) are generated as algebras by certain subsets of the usual algebra generators of these algebras. Note that the *normal* Hopf subalgebras have already been classified in the appendix of [1].

Proposition 6.1. *Let J_1, J_2 be finite index sets and $J := J_1 \dot{\cup} J_2$. Let H be a pointed Hopf algebra over k and $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) be subalgebras which are generated as algebras by elements t_j which are $(g_j, 1)$-primitive (in H, where $g_j \in G(H)$) for $j \in J_1$ (resp. $j \in J_2$). Define $V := \{t_j \mid j \in J\}$. Assume the following conditions:*

(a) *The map $\mathfrak{a}_1 \otimes \mathfrak{a}_2 \otimes G(H) \to H$, $a_1 \otimes a_2 \otimes g \mapsto a_1 a_2 g$ is a k-linear isomorphism.*

(b) *All $(g, 1)$-primitive elements (for $g \in G(H)$) are contained in the vector subspace of H spanned by $G(H)$ and V. For all $j \in J$ and $g \in G(H)$ we have $t_j g = \alpha_{g,j} g t_j$ where $\alpha_{g,j} \in k \setminus \{0\}$.*

(c) *For all $j \in J$ we have $\alpha_j := \alpha_{g_j, j} \neq 1$, and $g_i = g_j \Rightarrow \alpha_i \neq \alpha_j$ for distinct $i, j \in J$.*

(d) *For $s \in \{1, 2\}$, $\mathfrak{a}_s$ is an $\mathbb{N}_0[J_s]$-graded algebra, where the gradation is given by $|t_j| = j$ for $j \in J_s$ (that is, all relations between the algebra generators are generated by homogeneous ones). This induces a gradation on H given by $|a_1 a_2 g| := |a_1| + |a_2| \in \mathbb{N}_0[J]$ for homogeneous elements $a_1 \in \mathfrak{a}_1$, $a_2 \in \mathfrak{a}_2$. Denote the vector subspace of homogeneous elements of degree $\nu \in \mathbb{N}_0[J]$ in H by H_ν.*

Then the Hopf subalgebras H' are exactly the algebras generated by subsets $T \subset V$ and subgroups $G \subset G(H)$ such that $t_j \in T$ implies $g_j \in G$, as algebras. All finite-dimensional subbialgebras of H are Hopf subalgebras.

Proof. Obviously the mentioned algebras are Hopf subalgebras of H. We show that this list contains all of them. Let H' be a subbialgebra of H and $G = G(H')$. If H' is finite-dimensional, then the submonoid G of the group $G(H)$ is finite and a group. If H' is a Hopf subalgebra then G is a group, too. Let (H'_n) be the coradical filtration of H'. Now H' is pointed as a subbialgebra of H, whence H'_0 is spanned by group-like elements and H'_1 by G and (g, h)-primitive elements where $g, h \in G$ by the Taft–Wilson theorem [12] or [9, Theorem 5.4.1]. In particular, G contains g_j if $t_j \in H'$. Let $g, h \in G$ and let $s \in H'$ be (h, hg)-primitive. Then $h^{-1}s$ is

$(1,g)$-primitive in H' and also in H, whence

$$h^{-1}s = \lambda_0(g-1) + \sum_{j\in J:\, g_j=g} \lambda_j t_j$$

where λ_0, λ_j belong to k. Since H' contains $h^{-1}s$ and g, it also contains the elements

$$g^r h^{-1} s g^{-r} = \lambda_0(g-1) + \sum_{j\in J:\, g_j=g} \alpha_j^{-r}\lambda_j t_j$$

for all $r \in \mathbb{N}_0$. By assumption, the coefficients α_j therein are distinct and not equal to 1, whence this system of linear equations for $\lambda_j t_j$ can be solved uniquely and $t_j \in H'$ if $\lambda_j \neq 0$. Let $T = H' \cap V$. Hence H_1' is spanned by G and products of elements of T and G as vector space. Let $\tilde{H}$ be the subalgebra of H' generated by H_1'. Assume that there is a minimal positive integer s such that $H_s' \not\subset \tilde{H}$ and let $x \in A_s \setminus \tilde{H}$. Using the Taft–Wilson theorem, we can assume without loss of generality

$$\Delta(x) - x\otimes g - h\otimes x \in A_{s-1}\otimes A_{s-1}, \tag{21}$$

where $g, h \in G$. We can write x as a finite sum $x = \sum_{p=1}^{M} x_{p,\nu_p}\gamma_p$ such that for all p, $x_{p,\nu_p} \in \mathfrak{a}_1\mathfrak{a}_2 \setminus \{0\}$ is homogeneous of degree $\nu_p \in \mathbb{N}_0[J]$ and $\gamma_p \in G(H)$. We show first that $\gamma_p \in G$ for all p. For $\nu \in \mathbb{N}_0[J]$ let

$$y_\nu := \sum_{p:\ \nu_p=\nu} x_{p,\nu_p}\gamma_p, \qquad \text{hence } x = \sum_\nu y_\nu .$$

In this sum for y_ν we can assume $\gamma_p \neq \gamma_{p'}$ whenever $p \neq p'$. Now it follows from

$$\Delta H_{\nu'} \subset \sum_{\nu_1+\nu_2=\nu'} H_{\nu_1}\otimes H_{\nu_2} \tag{22}$$

for all $\nu' \in \mathbb{N}_0[J]$ that $\Delta(y_\nu)$ can be obtained from $\Delta(x)$ as the image of the projection of $H\otimes H$ to

$$\sum_{\nu_1+\nu_2=\nu} H_{\nu_1}\otimes H_{\nu_2}.$$

The contribution of $\Delta(x) - x\otimes g - h\otimes x$ to $H_\nu \otimes H_0$ is

$$\sum_{p:\ \nu_p=\nu} x_{p,\nu_p}\gamma_p\otimes\gamma_p - y_\nu\otimes g \text{ if } \nu\neq 0 \qquad \text{and}$$

$$\sum_{p:\ \nu_p=0} \gamma_p\otimes\gamma_p - \gamma_p\otimes g - h\otimes\gamma_p$$

otherwise and belongs to $\tilde{H}\otimes k[G]$ by (21). Thus $\gamma_p \in G$ for all p, using linear independence of distinct group-like elements. For each $\nu = \sum_{j\in J}\alpha_j j \in \mathbb{N}_0[J]$, call α_j the *j-coordinate* of ν. The elements of $\tilde{H}$ are sums of homogeneous elements of H; if the homogeneous summand in H_ν does not vanish then $t_j \notin T$ implies that the j-coordinate of ν vanishes. Since the group-like elements γ_p belong to $\tilde{H}$, there must be a homogeneous summand $y_\nu \neq 0$ with positive j-coordinate of ν and $t_j \notin T$. Now assume that y_ν is not (g,h)-primitive. Then $\Delta(y_\nu) - y_\nu \otimes g - h \otimes y_\nu$ contains non-zero homogeneous summands in

$$\sum_{\substack{\nu_1+\nu_2=\nu \\ \nu_1\neq 0,\, \nu_2\neq 0}} H_{\nu_1} \otimes H_{\nu_2},$$

where either ν_1 or ν_2 has positive j-component. Equation (22) implies that these summands cannot be killed by contributions of other homogeneous summands. But then (21) implies $H'_{s-1} \not\subset \tilde{H}$ in contrary to the minimality of s. Hence x is the sum of an element $x' \in \tilde{H}$ and homogeneous (g,h)-primitive summands, the sum of which being x''. Now $x'' = x - x'$ belongs to H', is (g,h)-primitive, and therefore belongs to H'_1, contrary to the definition of $\tilde{H}$. Hence $H' = \tilde{H}$. ∎

Remark 6.2. (a) There is an analogous proof of this proposition if H is as vector space isomorphic to the tensor product of more than two subalgebras and $G(H)$.

(b) If assumption (c) of the proposition is not satisfied then the above classification contains all Hopf subalgebras H' such that H'_1 (the second term of the coradical filtration containing all group-like and skew-primitive elements) is spanned by $G(H')$ and a subset $T \subset V$ as a vector space.

Now the Hopf subalgebras of the Frobenius–Lusztig kernels can be classified:

THEOREM 6.3. *Assume that q fits to $\tilde{\mathbf{f}}$ and $q^{2j\cdot j} \neq 1$ for all $j \in I$. Let $\tilde{G}$ be a group consisting of central group-like elements of $\tilde{\mathfrak{u}}$ such that $\tilde{K}_i\tilde{K}_j^{-1} \notin \tilde{G}$ for all $i, j \in I$ whenever $i \neq j$. Let $\tilde{u} := \tilde{\mathfrak{u}}/(\{g-1 \mid g \in \tilde{G}\})$. Elements of $\tilde{u}$ are denoted as in $\tilde{\mathfrak{u}}$. Then the Hopf subalgebras of $\tilde{u}$ are exactly those generated as algebras by a subset $T \subset \{E_j, F_j \mid j \in I\}$ and a subgroup $G \subset G(\tilde{u})$ such that*

$$\forall j \in I\colon \Big(E_j \in T \vee F_j \in T \Rightarrow \tilde{K}_j \in G\Big).$$

If $\tilde{u}$ is finite-dimensional then all subbialgebras are Hopf subalgebras and of this type.

Proof. We establish the assumptions of Proposition 6.1. $\tilde{u}$ is spanned by group-likes and $(1, \tilde{K}_i)$-primitive elements E_i, $\tilde{K}_iF_i$ for all $i \in I$ as algebra. The statement for the skew-primitive elements follows from [10, 3.1]; moreover the commutation relations with group-like elements are satisfied which implies (b). Together with the triangular decomposition, this gives (a), and the gradations on $\tilde{u}^+$ and $\tilde{u}^-$ yield (d). Moreover for all $j \in I$

$$\tilde{K}_j E_j \tilde{K}_j^{-1} = q^{j \cdot j} E_j, \qquad \tilde{K}_j \left(\tilde{K}_j F_j \right) \tilde{K}_j^{-1} = q^{-j \cdot j} \tilde{K}_j F_j.$$

The assumption on $\tilde{G}$ and $q^{2j \cdot j} \neq 1$ for all $j \in I$ imply assumption (c). ∎

THEOREM 6.4. *Let $(I, \cdot)$ be an arbitrary Cartan datum. If the elements $\tilde{K}_i$ in $\mathbf{U}$ for $i \in I$ are pairwise distinct, e.g., if the root datum is Y-regular* [7, 2.2.2], *then the Hopf subalgebras U of $\mathbf{U}$ are just those which are generated as algebras by a subset $T \subset \{E_j, F_j \mid j \in I\}$ and a subgroup $G \subset G(\mathbf{U})$ such that*

$$\forall j \in I\colon \left(E_j \in T \vee F_j \in T \Rightarrow \tilde{K}_j \in G \right).$$

Proof. We establish the assumptions of Proposition 6.1. $\mathbf{U}$ is generated by group-likes and $(1, \tilde{K}_i)$-primitive elements E_i, $\tilde{K}_i F_i$ for all $i \in I$ as algebra. The statement for the skew-primitive elements follows from [13]; moreover the commutation relations with group-like elements are satisfied which implies (b). Together with the triangular decomposition, this gives (a), and the gradations on $\mathbf{U}^+$ and $\mathbf{U}^-$ yield (d). Moreover for all $j \in I$

$$\tilde{K}_j E_j \tilde{K}_j^{-1} = v^{j \cdot j} E_j, \qquad \tilde{K}_j \left(\tilde{K}_j F_j \right) \tilde{K}_j^{-1} = v^{-j \cdot j} \tilde{K}_j F_j.$$

Now assumption (c) follows because v is not a root of unity. ∎

Remark 6.5. If the determinant of the Cartan matrix is coprime to the order of q, then it is easy to show that $\tilde{K}_i \tilde{K}_j^{-1}$ is not central for $i \neq j$ (cf. [10, 4.3 (c)]). Hence in this case there are no restrictions for $\tilde{G}$ in Theorem 6.3.

7. QUOTIENTS OF THE DUAL FROBENIUS–LUSZTIG KERNELS OF QUANTUM $GL(n)$ AND $SL(n)$

We recall some definitions from Sections 2 and 3 of [15], but we use opposite comultiplication for the quantized enveloping algebra in accordance with [6] and the other parts of this paper and opposite multiplication on the quantized matrix algebra which just means to replace q by its inverse. Let k be a field and $q \in k \setminus \{0\}$.

DEFINITION 7.1. Fix an integer $n > 1$, define a Cartan datum $(I, \cdot)$ by $I := \{1, \ldots, n-1\}$ and

$$i \cdot j = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and a root datum of type $(I, \cdot)$ by $X = Y = \mathbb{Z}^n$ and $\langle (a_j), (b_j) \rangle = \sum_{j=1}^n a_j b_j$ for $a_j, b_j \in \mathbb{Z}$, and the imbeddings $I \to X$, $I \to Y$ are given by

$$j \mapsto (0, \ldots, 0, \underset{\smile}{\overset{j}{1}}, \underset{\smile}{\overset{j+1}{-1}}, 0, \ldots, 0).$$

The quantized enveloping algebra attached to this root datum will be called $U(GL_q(n))$; the Hopf subalgebra generated by E_i, F_i, $K_i^{\pm 1}$ for $i \in I$ will be called $U(SL_q(n))$. The *quantized matrix algebra* $A(M_q(n))$ is generated by the elements α_{ij} as algebra for $1 \leq i, j \leq n$, subject to the relations

$$\alpha_{is}\alpha_{js} = q\alpha_{js}\alpha_{is}, \qquad \alpha_{si}\alpha_{sj} = q\alpha_{sj}\alpha_{si} \text{ if } i < j,$$
$$\alpha_{it}\alpha_{js} = \alpha_{js}\alpha_{it}, \quad \alpha_{is}\alpha_{jt} - \alpha_{jt}\alpha_{is} = \left(q - q^{-1}\right)\alpha_{it}\alpha_{js} \text{ if } i < j, s < t.$$

A bialgebra structure is given by

$$\Delta(\alpha_{ij}) = \sum_{s=1}^n \alpha_{is} \otimes \alpha_{sj}, \qquad \varepsilon(\alpha_{ij}) = \delta_{ij}$$

for $1 \leq i, j \leq n$. Let $s \leq n$, $\mathbf{I} := \{i_1, i_2, \ldots, i_s\}$, and $\mathbf{J} := \{j_1, j_2, \ldots, j_s\}$ where $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq n$. Define the *quantum subdeterminants*

$$\det{}_{\mathbf{J}}^{\mathbf{I}} := \sum_{\sigma \in \mathrm{Sym}(s)} (-q)^{l(\sigma)} \alpha_{i_1 j_{\sigma(1)}} \alpha_{i_2 j_{\sigma(2)}} \cdots \alpha_{i_s j_{\sigma(s)}}$$

(for each permutation $\sigma \in \mathrm{Sym}(s)$ let $l(\sigma)$ denote the length of σ). In particular define $X_{ij} := \det_{\mathbf{I}}^{\mathbf{I}}$ for $\mathbf{I} = \{i, i+1, \ldots, j\}$ and $\det_q := X_{1n}$, the *quantum determinant*. The localization $A(GL_q(n)) := A(M_q(n))[\det_q^{-1}]$ and the quotient $A(SL_q(n)) := A(GL_q(n))/(\det_q - 1)$ are Hopf algebras. Their elements will be denoted like the elements of $A(M_q(n))$. There is a Hopf pairing $\langle\, ,\rangle$: $U(GL_q(n)) \times A(GL_q(n)) \to k$ which induces a Hopf pairing of $U(SL_q(n))$ and $A(SL_q(n))$ and is given on the algebra generators by

$$\langle K_{(a_t)}, \alpha_{ij} \rangle = \delta_{ij} q^{a_i}, \qquad \langle E_s, \alpha_{ij} \rangle = \delta_{si}\delta_{s+1,j}, \qquad \langle F_s, \alpha_{ij} \rangle = \delta_{s+1,i}\delta_{sj}, \tag{23}$$

for $(a_t) \in Y$, $s \in I$, $1 \leq i, j \leq n$.

We can now define the dual Frobenius–Lusztig kernels.

DEFINITION 7.2. Let q be a root of unity of order N and let ℓ be the order of q^2. Define

$$\tilde{a} := A\big(GL_q(n)\big)/\Big(\Big\{\alpha_{ss}^N - 1, \alpha_{ij}^{\ell} \,|\, i \neq j\Big\}\Big),$$

$$\tilde{a}' := A\big(SL_q(n)\big)/\Big(\Big\{\alpha_{ss}^N - 1, \alpha_{ij}^{\ell} \,|\, i \neq j\Big\}\Big),$$

where $s, i, j \in \{1, \dots, n\}$. Let $\mathfrak{u}$ and $\mathfrak{u}'$ be the Frobenius–Lusztig kernels of $U(GL_q(n))$ and $U(SL_q(n))$. Define

$$\tilde{u} := \mathfrak{u}/\Big(\Big\{K_\mu^N - 1 \,|\, \mu \in Y\Big\}\Big),$$

$$\tilde{u}' := \mathfrak{u}'/\Big(\Big\{K_i^N - 1 \,|\, i \in I\Big\}\Big).$$

Note that $\tilde{u}, \tilde{u}'$ correspond to $U^{[\ell]}(GL_q(n))$ and $U^{[\ell]}(SL_q(n))$ from [15, Chap. 12], respectively.

Then we have by [14, Proposition 6.3]:

THEOREM 7.3. *The Hopf pairing from* (23) *induces isomorphisms* $\tilde{u} \cong \tilde{a}^*$ *and* $\tilde{u}' \cong (\tilde{a}')^*$.

Assume that q is a root of unity and $q^4 \neq 1$. Now we interpret the duals of the Hopf subalgebras of $\tilde{u}$ and $\tilde{u}'$ as factor Hopf algebras of $\tilde{a}$ and $\tilde{a}'$.

THEOREM 7.4. *Let* $\mathcal{T} := \{E_i, F_i \,|\, i \in I\}$. *For each subset* $T \subset \mathcal{T}$ *define*

$$W_T = \{i \in I \,|\, E_i \in T \vee F_i \in T\},$$

$$P_T = \Big\{\alpha_{ij} \,|\, \exists h \in I\colon (E_h \notin T \wedge i \leq h < j) \vee (F_h \notin T \wedge j \leq h < i)\Big\},$$

$$G_T = \Big\{X_{ij} \,|\, 1 \leq i \leq j \leq n, i - 1 \notin W_T \wedge j \notin W_T\Big\}.$$

Then each bi-ideal $\mathcal{I}$ *of* $\tilde{a}$ *is a Hopf ideal and there is a subset* $T \subset \mathcal{T}$ *and a set* G' *of monomials in* G_T *such that* $\mathcal{I}$ *is generated by* P_T *and* $\{g - 1 \,|\, g \in G'\}$ *as a two-sided ideal.*

Proof. Clearly the assumptions of Theorem 6.3 are satisfied. All sub-bialgebras of the finite-dimensional Hopf algebra $\tilde{u}$ are Hopf algebras, whence by Theorem 7.3 all bi-ideals are Hopf ideals. For each bi-ideal $\mathcal{I}$ of $\tilde{a}$ define $T = \mathcal{T} \cap (\tilde{a}/\mathcal{I})^*$. For fixed T denote the two-sided ideal of $\tilde{a}$ generated by P_T as $\mathcal{I}_T$. Direct calculation shows that $\mathcal{I}_T$ is a bi-ideal (and Hopf ideal). The rest of the proof consists of several steps.

(a) The canonical projection $\tilde{a} \to \tilde{a}/\mathscr{I}_T$ maps the elements of G_T to central group-like elements.

Proof. Fix $r, s \in \{0, \dots, n\} \setminus W_T$, $r \le s$. Define $M_1 = \{1, \dots, r\}$, $M_2 = \{r+1, \dots, s\}$, $M_3 = \{s+1, \dots, n\}$. Consider the "blocks" consisting of the elements α_{ij}, where i, j are both in M_1 or in M_2 or in M_3. All other algebra generators vanish in $\tilde{a}/\mathscr{I}_T$, whence elements of different blocks commute in $\tilde{a}/\mathscr{I}_T$ and the subalgebras generated by all elements of a block are isomorphic to quotients of quantized matrix algebras, whence $\overline{X}_{1,r}$, $\overline{X}_{r+1,s}$, and $\overline{X}_{s+1,n}$ are central group-like elements of $\tilde{a}/\mathscr{I}_T$.

(b) All ideals mentioned above are Hopf ideals, because by part (a), the corresponding factor algebras are quotients of the Hopf algebra $\tilde{a}/\mathscr{I}_T$ by central group-like elements.

(c) The Hopf subalgebra $(\tilde{a}/(P_T \cup \{g-1 \mid g \in G'\}))^*$ of $\tilde{u}$ contains T and all elements K_h where $h \in W_T$.

Proof. Using the duality in Theorem 7.3, it suffices to prove that for any element x of these and any element y of the ideal we have $\langle x, y\rangle = 0$. Since x is skew-primitive or group-like, it suffices to prove this on the generators of the ideal. We only discuss the most difficult computation: $\langle x, X_{r+1,s} - 1\rangle = 0$ for $r, s \in \{0, \dots, n\} \setminus W_T$. We have

$$\begin{aligned}\langle K_h, X_{r+1,s} - 1\rangle &= \langle K_h, \alpha_{r+1,r+1}\rangle \cdots \langle K_h, \alpha_{ss}\rangle - 1 \\ &= q^{\delta_{h+1,r+1} - \delta_{h,s}} - 1 = 0.\end{aligned}$$

Moreover $\langle E_h, X_{r+1,s} - 1\rangle = \langle E_h, X_{r+1,s}\rangle = 0$ because $X_{r+1,s}$ consists of the summand $\alpha_{r+1,r+1} \cdots \alpha_{ss}$ each factor of which being in the kernel of $\langle E_h, \cdot\,\rangle$ and summands with at least two elements which are not in the diagonal and hence are in the kernels of $\langle K_h, \cdot\,\rangle$ and $\langle 1, \cdot\,\rangle$. A similar proof shows $\langle F_h, X_{r+1,s}\rangle = 0$.

(d) We have $\mathscr{T} \cap (\tilde{a}/\mathscr{I}_T)^* = T$, because the inclusion $\supset$ has been shown in part (c), and $\subset$ is trivial (because $\langle E_i, \alpha_{i,i+1}\rangle = 1 = \langle F_i, \alpha_{i+1,i}\rangle$).

(e) We have $G(\tilde{u}) \subset (\tilde{a}/\mathscr{I}_T)^*$, since the maps $\langle K_\mu, \cdot\,\rangle$ for $\mu \in Y$ are characters and $\langle K_\mu, \alpha_{ij}\rangle = 0$ for $\alpha_{ij} \in P_T$.

(f) For each $g \in G_T$, the order of $g + \mathscr{I}_T$ in $\tilde{a}/\mathscr{I}_T$ is N.

Proof. By (e), $(\tilde{a}/\mathscr{I}_T)^*$ contains all group-like elements of $\tilde{u}$, and $\tilde{a}/\mathscr{I}_T$ is dual to the subalgebra of $\tilde{u}$ generated by T and $G(\tilde{u})$, because this is the biggest Hopf subalgebra H of $\tilde{u}$ such that $\mathscr{T} \cap H = T$ (by Theorem 6.3). Fix $r, s \in \{0, \dots, n\} \setminus W_T$, where $r \le s$. Since $\tilde{a}/\mathscr{I}_T$ is dual to the Hopf subalgebra H of $\tilde{u}$ generated by T and $G(\tilde{u})$ as algebra, we have to compute the smallest positive integer z such that $\langle x, X_{r+1,s}^z - 1\rangle = 0$ or $\langle x, X_{r+1,s}^z\rangle = \varepsilon(x)$ for all $x \in H$. Since $\overline{X}_{r+1,s}^z$ is group-like, the linear form $\langle \cdot, X_{r+1,s}^z\rangle$ is a character on H, and it suffices to prove the assertion for algebra generators of H. We have $\langle E_i, X_{r+1,s}^z\rangle = \varepsilon(E_i), \langle F_i, X_{r+1,s}^z\rangle = \varepsilon(F_i)$ for all $i \in I$ by part (c). For all $(a_j) \in Y$ and $t \in \mathbb{N}$ we have

$$\langle K_{(a_j)}, X_{r+1,s}^t\rangle = \left(\langle K_{(a_j)}, \alpha_{r+1,r+1}\rangle \cdots \langle K_{(a_j)}, \alpha_{ss}\rangle\right)^t = q^{t(a_{r+1}+\cdots+a_s)}.$$

Hence the order is N.

(g) Let $\mathscr{I}$ be a Hopf ideal of $\tilde{a}$ and let $T := \mathscr{T} \cap (\tilde{a}/\mathscr{I})^*$. It follows from Theorem 6.3 that $(\tilde{a}/\mathscr{I})^* \subset \tilde{u}$ is generated by T and group-like elements as algebra. Assume

$$\{0, \ldots, n\} \setminus W_T = \{r(1), r(2), \ldots, r(s+1)\}$$

where $0 = r(1) < r(2) < \cdots < r(s+1) = n$. Define $e_j = (0, \ldots, 1, \ldots, 0) \in Y$ (1 at the jth coordinate) for $1 \le j \le n$. Since $(\tilde{a}/\mathscr{I})^*$ contains all elements $K_i = K_{e_i}K_{e_{i+1}}^{-1}$ where $i \in W_T$, these subgroups of $G((\tilde{a}/\mathscr{I})^*)$ correspond exactly to the subgroups of the group R_u generated by $K_{e_{r(1)+1}}, K_{e_{r(2)+1}}, \ldots, K_{e_{r(s)+1}}$. By definition of $\tilde{u}$, R_u is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^s$. Since there is a one-to-one correspondence between Hopf subalgebras of $\tilde{u}$ and factor Hopf algebras of $\tilde{a}$, it suffices to prove that there are no more than the Hopf algebras described in (c). By part (f), the central group-likes $X_{r(t)+1,r(t+1)} + \mathscr{I}_T$, where $1 \le t \le s$ in $\tilde{a}/\mathscr{I}_T$, have order N; they generate a group called R_a. The pairing between $\tilde{u}$ and $\tilde{a}$ induces a pairing between R_u and R_a given by the bi-character

$$\langle K_{e_{r(i)+1}}, X_{r(j)+1,r(j+1)} + \mathscr{I}_T\rangle = q^{\delta_{ij}}.$$

Since q has order N, there are no more relations between the commuting generators $X_{r(t)+1,r(t+1)} + \mathscr{I}_T$, and R_a is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^s$. This duality gives a one-to-one correspondence between the subgroups of the finite group R_u and the quotients of R_a. Hence we have found all factor Hopf algebras of $\tilde{a}$.

COROLLARY 7.5. *Since $\tilde{a}'$ is a factor Hopf algebra of $\tilde{a}$, the factor Hopf algebras of $\tilde{a}'$ are exactly those, whose quantum determinant is* 1.

We determine the central group-like elements of these factor Hopf algebras.

THEOREM 7.6. *Let $\mathscr{I}$ be a Hopf ideal of $\tilde{a}$ and $T := \mathscr{T} \cap (\tilde{a}/\mathscr{I})^*$. The central group-like elements of $\tilde{a}/\mathscr{I}$ are exactly the elements*

$$X_{r(1)+1,r(2)}^{\beta_1} \cdots X_{r(s)+1,r(s+1)}^{\beta_s} + \mathscr{I},$$

where $\{0, \ldots, n\} \setminus W_T = \{r(1), r(2), \ldots, r(s+1)\}$, *where* $0 = r(1) < r(2) < \cdots < r(s+1) = n$, *and* $\beta_1, \ldots, \beta_s \in \mathbb{N}$.

Proof. Obviously the mentioned elements are central and group-like. Their number is the order of R'_a, a quotient of R_a from Theorem 7.4(g). Let R'_u be the corresponding subgroup of R_u from Theorem 7.4(g). Central group-like elements of $\tilde{a}/\mathscr{I}$ correspond to algebra homomorphisms ϕ: $\tilde{u} \to k$ such that for all $\psi \in \operatorname{Hom}(\tilde{u}, k)$: $\psi * \phi = \phi * \psi$. This gives $\phi(x) = 0$ for $x \in T$ and $\phi(\tilde{K}_i) = 1$ if $E_i \in T$ or $F_i \in T$. Therefore these homomorphisms correspond to characters of R'_u. Their number is the order of R'_u, because R'_u is finite and abelian. Finally it follows from $|R'_u| = |R'_a|$ that there are no central group-like elements in addition to the elements mentioned above. ∎

8. QUASITRIANGULAR STRUCTURES

Let q be a root of unity. The aim of this section is to construct a Quasi-$\mathscr{R}$-matrix for the Frobenius–Lusztig kernels imitating the procedure in [6, Chap. 4]. The Quasi-$\mathscr{R}$-matrix gives rise to quasitriangular structures. In particular, all quasitriangular structures from the classification in [4] will be described explicitly. Let $\tilde{\mathfrak{u}}^{\diamond}$ be the Frobenius–Lusztig kernel obtained for q^{-1} instead of q. Then $\tilde{\mathfrak{u}}$ and $\tilde{\mathfrak{u}}^{\diamond}$ can be identified as sets.

DEFINITION 8.1. There is a unique isomorphism of algebras

$$^{-}: \tilde{\mathfrak{u}} \to \tilde{\mathfrak{u}}^{\diamond}, E_i \mapsto E_i, \qquad F_i \mapsto F_i, \qquad K_\mu \mapsto K_{-\mu}$$

for all $i \in I$, $\mu \in Y$ [6, 3.1.12]; the map $^{-}\otimes{}^{-}: \tilde{\mathfrak{u}} \otimes \tilde{\mathfrak{u}} \to \tilde{\mathfrak{u}}^{\diamond} \otimes \tilde{\mathfrak{u}}^{\diamond}$ is a well-defined linear algebra isomorphism. Then we can define in analogy to [7, 4.1.1], $\overline{\Delta}$: $\tilde{\mathfrak{u}} \to \tilde{\mathfrak{u}} \otimes \tilde{\mathfrak{u}}, x \mapsto (^{-}\otimes{}^{-})\Delta(\bar{x})$.

8.1. *The Quasi-$\mathscr{R}$-Matrix of* $\tilde{\mathfrak{u}}$

Assume that $(I, \cdot)$ is of finite type, that q fits to $\tilde{\mathbf{f}}$, and $q_i^2 \neq 1$ for all $i \in I$. We take the original definition $B_i = (1 - q_i^{-2})^{-1}$. The proof of the following theorem can be repeated word by word (after replacing v by q) from [6, Theorem 4.1.2], because it only uses that $(\cdot, \cdot)$ is non-degenerate (cf. [10, 2.16]).

THEOREM 8.2. (a) *There is a unique family of elements* $\Theta_\nu \in \tilde{\mathfrak{u}}_\nu^- \otimes \tilde{\mathfrak{u}}_\nu^+$ (*with* $\nu \in \mathbb{N}_0[I]$) *such that* $\Theta_0 = 1 \otimes 1$ *and* $\Theta = \sum_\nu \Theta_\nu$ *satisfies* $\Delta(x)\Theta = \Theta\overline{\Delta}(x)$ *for all* $x \in \tilde{\mathfrak{u}}$ (*identity in* $\tilde{\mathfrak{u}} \otimes \tilde{\mathfrak{u}}$).

(b) *Let B be a k-basis of $\tilde{\mathbf{f}}$ such that $B_\nu = B \cap \tilde{\mathbf{f}}_\nu$ is a basis of $\tilde{\mathbf{f}}_\nu$ for any ν. Let $\{b^* \mid b \in B_\nu\}$ be the basis of $\tilde{\mathbf{f}}_\nu$ dual to B_ν under $(\cdot, \cdot)$. We have*

$$\Theta_\nu = (-1)^{\operatorname{tr}\nu} q_\nu \sum_{b \in B_\nu} b^- \otimes b^{*+} \in \tilde{\mathfrak{u}}_\nu^- \otimes \tilde{\mathfrak{u}}_\nu^+,$$

where $q_\nu = \prod_i q_i^{\nu_i}$ *for* $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$.

Remark 8.3. (a) The element Θ is called Quasi-$\mathscr{R}$-matrix (cf. [6, Paragraph 4.1.4]).

(b) Since Θ is unique, it is independent of the actual choice of basis.

(c) Since $\tilde{\mathbf{f}}$ is finite-dimensional, it is not necessary to consider a completion of $\tilde{\mathfrak{u}} \otimes \tilde{\mathfrak{u}}$ with respect to a certain topology (cf. [6, Paragraph 4.1.1]).

Here are some properties of the Quasi-$\mathscr{R}$-matrix. For an element $P = \sum_t x_t \otimes y_t \in V \otimes V$, where V is a vector space, we shall denote the elements $\sum_t x_t \otimes y_t \otimes 1, \sum_t 1 \otimes x_t \otimes y_t, \sum_t x_t \otimes 1 \otimes y_t$ by P^{12}, P^{23}, P^{13}, respectively [6, Paragraph 4.2.1].

PROPOSITION 8.4. *We have* $\Theta\overline{\Theta} = \overline{\Theta}\Theta = 1 \otimes 1$ *in* $\tilde{\mathfrak{u}} \otimes \tilde{\mathfrak{u}}$.

Proof. Compare [6, Corollary 4.1.3]. ∎

PROPOSITION 8.5. *For all* $\nu \in \mathbb{N}_0[I]$ *we have*

$$(\Delta \otimes \mathrm{id})\overline{\Theta}_\nu = \sum_{\nu' + \nu'' = \nu} \overline{\Theta}^{13}_{\nu'}\left(1 \otimes \tilde{K}_{-\nu'} \otimes 1\right)\overline{\Theta}^{23}_{\nu''},$$

$$(\mathrm{id} \otimes \Delta)\overline{\Theta}_\nu = \sum_{\nu' + \nu'' = \nu} \overline{\Theta}^{13}_{\nu'}\left(1 \otimes \tilde{K}_{\nu'} \otimes 1\right)\overline{\Theta}^{12}_{\nu''}.$$

Proof. This is similar to [6, Proposition 4.2.3]. ∎

Remark 8.6. The lemma in [10, 2.9] contains a more detailed description. Reference [10, Eq. (7)] gives

$$\Theta = \sum_{\mathbf{c}} (-1)^{\mathrm{tr}|b_{\mathbf{c}}|} q_{|b_{\mathbf{c}}|} b^-_{\mathbf{c}} \otimes \tilde{b}^+_{\mathbf{c}}.$$

The Quasi-$\mathscr{R}$-matrix is invariant under Hopf algebra automorphisms:

PROPOSITION 8.7. *Assume that there are no weak diagram antimorphisms of* $(I, \cdot)$ (*e.g., if it is irreducible*). *Let* $\phi\colon \tilde{\mathfrak{u}} \to \tilde{\mathfrak{u}}$ *be an automorphism. Then* $(\phi \otimes \phi)(\Theta) = \Theta, (\phi \otimes \phi)\overline{\Theta} = \overline{\Theta}$.

Proof. It follows from [10, 5.14] and the only exceptional case in [10, 5.15] that irreducible Cartan data of finite type do not admit weak diagram antimorphisms. By [10, 5.14] there are coefficients $a_i \in k \setminus \{0\}$ for $i \in I$ and a weak diagram automorphism σ of I such that

$$\phi(E_i) = a_i E_{\sigma(i)}, \qquad \phi(F_i) = q_i q^{-1}_{\sigma(i)} a_i^{-1} F_{\sigma(i)}, \qquad \phi\left(\tilde{K}_i\right) = \tilde{K}_{\sigma(i)}.$$

In particular, $(\phi \otimes \phi)(\Theta) \in \tilde{\mathfrak{u}}^- \otimes \tilde{\mathfrak{u}}^+$. If we can prove that

$$\overline{\Delta}\phi(x) = (\phi \otimes \phi)\overline{\Delta}(x) \tag{24}$$

for all $x \in \tilde{\mathfrak{u}}$, we have $\Delta(\phi(x))(\phi \otimes \phi)(\Theta) = (\phi \otimes \phi)(\Theta)\overline{\Delta}\phi(x)$, and the uniqueness of Θ in Theorem 8.2 implies the assertion for Θ and $\overline{\Theta}$, because $\overline{\Theta}$ is the inverse of Θ. Now we prove (24). Since the maps on both sides are k-algebra homomorphisms, it suffices to check (24) on the algebra generators. We have

$$\overline{\Delta}\phi(E_i) = a_i\Big(E_{\sigma(i)} \otimes 1 + \tilde{K}_{\sigma(i)}^{-1} \otimes E_{\sigma(i)}\Big) = (\phi \otimes \phi)\overline{\Delta}(E_i),$$

and a similar computation holds for F_i. For each $\mu \in Y$ we have $\overline{\Delta}(K_\mu) = K_\mu \otimes K_\mu$, whence (24) holds for $x = K_\mu$. ∎

Remark 8.8. The proof of Proposition 8.7 can be applied to the situation of the Quasi-$\mathscr{R}$-matrix of **U**, because the automorphisms of **U** are known (cf. [2, Theorem C]).

8.2. *Quasitriangular Structures*

We recall the definition of a quasitriangular structure.

DEFINITION 8.9. Let H be a Hopf algebra. An invertible element R of $H \otimes H$ is called *quasitriangular structure*, if

$$\Delta^{\text{op}}(x)R = R\Delta(x) \text{ for all } x \in H, \qquad (\Delta \otimes \text{id})R = R^{13}R^{23},$$
$$(\text{id} \otimes \Delta)R = R^{13}R^{12}. \tag{25}$$

Notation 8.10. (a) For $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$ let $\lceil \nu \rfloor = \sum_i \nu_i \frac{i \cdot i}{2} i$.

(b) In the Frobenius–Lusztig kernels there are some extra relations between the group-like elements, namely $\tilde{K}_i^{2l_i} = 1$ for $i \in I$, where l_i is the order of q_i^2. These are all relations (cf. [7, Theorem 8.3(ii)]). Let $Y' \subset Y$ be the subgroup generated by $2l_i \lceil i \rfloor$ for $i \in I$. Define $\tilde{Y} = Y/Y'$.

THEOREM 8.11. *Let $G \subset \{\mu \in \tilde{Y} \mid K_\mu$ central$\}$ be a subgroup. Let Y_1, Y_2 be finite subgroups of $\tilde{Y}/G$ which contain the images of $\lceil \nu \rfloor \in Y$. In the following $\mu, \tilde{\mu}, \mu_1, \mu_2$ and $\rho, \tilde{\rho}, \rho_1, \rho_2$ denote elements of Y_1 and Y_2, respectively. The element $R = R_0\overline{\Theta}$ where $R_0 = \sum_{\mu,\rho} f(\mu,\rho)K_\mu \otimes K_\rho$ is a quasitriangular structure of $\tilde{\mathfrak{u}} = \hat{\mathfrak{u}}/(\{K_\mu - 1 \mid \mu \in G\})$ if and only if for all $i \in I$ and $\mu, \rho, \tilde{\mu}, \tilde{\rho}$,*

$$f(\mu + \lceil i \rfloor, \rho) = f(\mu,\rho)q^{-\langle \rho, i' \rangle}, \qquad f(\mu, \rho + \lceil i \rfloor) = f(\mu,\rho)q^{-\langle \mu, i' \rangle} \tag{26}$$

$$\sum_{\substack{\rho_1,\rho_2 \in Y_2 \\ \rho_1+\rho_2=\rho}} f(\mu,\rho_1)f(\tilde{\mu},\rho_2) = \delta_{\mu,\tilde{\mu}} f(\mu,\rho),$$

$$\sum_{\substack{\mu_1,\mu_2 \in Y_1 \\ \mu_1+\mu_2=\mu}} f(\mu_1,\rho)f(\mu_2,\tilde{\rho}) = \delta_{\rho,\tilde{\rho}} f(\mu,\rho), \tag{27}$$

$$\sum_\mu f(\mu,\rho) = \delta_{\rho,0} \vee \sum_\rho f(\mu,\rho) = \delta_{\mu,0}. \tag{28}$$

Condition (28) *follows from* (26) *and* (27) *if there exists* $l \in k \setminus \{0\}$ *such that* $f(\mu, 0) = f(0, \rho) = l$ *for all* μ, ρ. *There are conditions on the order of* q: *For all* μ, ρ *for which there exist* $\tilde{\mu}, \tilde{\rho}$ *such that* $f(\mu, \tilde{\rho}) \neq 0, f(\tilde{\mu}, \rho) \neq 0$ *we have*

$$q^{2l_i\langle \mu, i'\rangle} = q^{2l_i\langle \rho, i'\rangle} = 1. \tag{29}$$

If this condition is satisfied then f *is well-defined on the preimages of* $Y_1 \times Y_2$ *under* $\tilde{Y} \to \tilde{Y}/G$.

Proof. It follows from (26) that $f(\mu + 2l_i\lceil i \rfloor, \rho) = q^{-2l_i\langle \rho, i'\rangle} f(\mu, \rho)$. These are the only relations in $\tilde{\mathfrak{u}}$ between the commuting group-like elements. We check the conditions in Definition 8.9.

(a) $\Delta^{\mathrm{op}}(x)R = R\Delta(x)$ or $\Delta^{\mathrm{op}}(x)R_0 = R_0\overline{\Delta}(x)$ for all $x \in \tilde{u}$. The set of elements of $\tilde{u}$ which satisfy this condition is a subalgebra of $\tilde{u}$ containing all K_μ. Hence, in order for this set to be equal to $\tilde{u}$, it is necessary and sufficient that it contains E_i, F_i for all $i \in I$. We check the condition for $x = E_i$.

$$\left(1 \otimes E_i + E_i \otimes \tilde{K}_i\right)R_0 = R_0\left(E_i \otimes 1 + \tilde{K}_{-i} \otimes E_i\right).$$

With respect to the gradation $\tilde{u}_{(\nu, \gamma)} = t(\tilde{\mathbf{f}}_\nu \otimes \tilde{u}^0 \otimes \tilde{\mathbf{f}}_\gamma)$, the elements $(1 \otimes E_i)R_0$ and $R_0(\tilde{K}_{-i} \otimes E_i)$ belong to $\tilde{u}_{(0,0)} \otimes \tilde{u}_{(i,0)}$ and $(E_i \otimes \tilde{K}_i)R_0, R_0(E_i \otimes 1)$ to $\tilde{u}_{(i,0)} \otimes \tilde{u}_{(0,0)}$. These vector subspaces of $\tilde{u} \otimes \tilde{u}$ intersect in $\{0\}$, whence the condition is equivalent to $(1 \otimes E_i)R_0 = R_0(\tilde{K}_{-i} \otimes E_i)$ and $(E_i \otimes \tilde{K}_i)R_0 = R_0(E_i \otimes 1)$. Now

$$\begin{aligned}(1 \otimes E_i)R_0 &= \sum_{\mu, \rho} f(\mu, \rho)q^{-\langle \rho, i'\rangle}\left(K_{\mu + \lceil i \rfloor} \otimes K_\rho\right)\left(\tilde{K}_{-i} \otimes E_i\right)\\ &\overset{!}{=} \sum_{\mu, \rho} f(\mu + \lceil i \rfloor, \rho)\left(K_{\mu + \lceil i \rfloor} \otimes K_\rho\right)\left(\tilde{K}_{-i} \otimes E_i\right) = R_0\left(\tilde{K}_{-i} \otimes E_i\right)\end{aligned}$$

holds if and only if $f(\mu + \lceil i \rfloor, \rho) = f(\mu, \rho)q^{-\langle \rho, i'\rangle}$ for all μ, ρ. A similar argument shows that $(E_i \otimes \tilde{K}_i)R_0 = R_0(E_i \otimes 1)$ is equivalent to $f(\mu, \rho + \lceil i \rfloor) = f(\mu, \rho)q^{-\langle \mu, i'\rangle}$ for all μ, ρ. The equations for F_i give the same conditions for f, with i replaced by $-i$, which are not really new.

(b) $(\Delta \otimes \mathrm{id})R = R^{13}R^{23}$. We have

$$(\Delta \otimes \mathrm{id})R = \sum_{\nu_1, \nu_2; \mu, \rho} f(\mu, \rho)\left(K_\mu \otimes K_{\mu - \lceil \nu_1 \rfloor} \otimes K_\rho\right)\overline{\Theta}^{13}_{\nu_1}\overline{\Theta}^{23}_{\nu_2}$$

and (summation over $\nu_1, \nu_2, \mu_1, \mu_2, \rho_1, \rho_2$)

$$\begin{aligned}R^{13}R^{23} &= \sum f(\mu_1, \rho_1)f(\mu_2, \rho_2)\left(K_{\mu_1} \otimes K_{\mu_2} \otimes K_{\rho_1 + \rho_2}\right)q^{-\langle \rho_2, \nu_1'\rangle}\overline{\Theta}^{13}_{\nu_1}\overline{\Theta}^{23}_{\nu_2}\\ &\underset{(26)}{=} \sum f(\mu_1, \rho_1)f(\mu_2 + \lceil \nu_1 \rfloor, \rho_2)\left(K_{\mu_1} \otimes K_{\mu_2} \otimes K_{\rho_1 + \rho_2}\right)\overline{\Theta}^{13}_{\nu_1}\overline{\Theta}^{23}_{\nu_2}\\ &= \sum f(\mu_1, \rho_1)f(\mu_2, \rho_2)\left(K_{\mu_1} \otimes K_{\mu_2 - \lceil \nu_1 \rfloor} \otimes K_{\rho_1 + \rho_2}\right)\overline{\Theta}^{13}_{\nu_1}\overline{\Theta}^{23}_{\nu_2}.\end{aligned}$$

Since the group-like elements are linearly independent, we have equality if and only if for all ρ, μ, μ',

$$\sum_{\rho_1+\rho_2=\rho} f(\mu, \rho_1) f(\mu', \rho_2) = \delta_{\mu,\mu'} f(\mu, \rho).$$

The other condition in (27) is equivalent to $(\mathrm{id} \otimes \Delta)R = R^{13}R^{12}$.

(c) R is invertible. Assuming the conditions (25) for R, we have the following equivalent conditions:

(i) R is invertible

(ii) $(\varepsilon \otimes \mathrm{id})R = 1$,

(iii) $(\mathrm{id} \otimes \varepsilon)R = 1$.

Proof. If R is invertible, R is a quasitriangular structure, and [5, Theorem VIII.2.4] implies (ii) and (iii), since the antipode of $\tilde{u}$ is clearly invertible. It follows from (ii) (or (iii)) that $(S \otimes \mathrm{id})R$ (or $(\mathrm{id} \otimes S^{-1})R$) is an inverse of R (cf. proof of [5, Theorem VIII.2.4]). We have $(\varepsilon \otimes \mathrm{id})\overline{\Theta} = (\mathrm{id} \otimes \varepsilon)\overline{\Theta} = 1$ because the only non-zero contribution comes from $\overline{\Theta}_0 = 1 \otimes 1$. Direct calculation shows that $(\mathrm{id} \otimes \varepsilon)R = (\varepsilon \otimes \mathrm{id})R = 1$ is equivalent to (28). If there exists $l \in k \setminus \{0\}$ such that $f(\mu, 0) = f(0, \rho) = l$ for all μ, ρ then (27) implies for $\mu' = 0$,

$$\sum_{\rho_1+\rho_2=\rho} f(\mu, \rho_1) l = \delta_{\mu,0} f(\mu, \rho) = \delta_{\mu,0} l \Leftrightarrow \sum_{\rho_1} f(\mu, \rho_1) = \delta_{\mu,0}.$$

∎

Remark 8.12. Conditions (27) and (28) just mean that R_0 is a quasitriangular structure of the group algebra $k[\{K_\mu \mid \mu \in \tilde{Y}\}]$ by [8]. Then (26) is a compatibility condition.

Note. It is always possible to choose y_1, y_2 such that $y_2 = y_1$ and $|y_1| f: y_1 x y_1 \to k \setminus \{0\}$ is a bicharacter.

Another class of quasitriangular structures can be constructed from the above.

PROPOSITION 8.13. (a) *If a Hopf algebra H admits a quasitriangular structure $R \in H \otimes H$ then $\tau(R^{-1})$ is a quasitriangular structure, too, where τ: $H \otimes H \to H \otimes H$, $x \otimes y \mapsto y \otimes x$ denotes the usual twist.*

(b) *Let $R_0 = \sum f(\mu, \rho) K_\mu \otimes K_\rho$ as in Theorem* 8.11. *Then $R_0^{-1} = \sum f(\mu, -\rho) K_\mu \otimes K_\rho = \sum f(-\mu, \rho) K_\mu \otimes K_\rho$. In particular, $f(\mu, \rho) = f(-\mu, -\rho)$ for all μ, ρ.*

(c) *If f satisfies* (26)–(28) *then* $\tilde{u}$ *admits the quasitriangular structures*

$$R_0\overline{\Theta} = \left(\sum_{\mu,\rho} f(\mu,\rho)\left(K_\mu \otimes K_\rho\right)\right)\left(\sum_\nu (-1)^{\operatorname{tr}\nu} q_\nu^{-1} \sum \bar{b}^- \otimes \left(\overline{b^*}\right)^+\right)$$

$$\tau(\Theta)\tau\left(R_0^{-1}\right) = \left(\sum_\nu (-1)^{\operatorname{tr}\nu} q_\nu \sum b^+ \otimes b^{*-}\right)\left(\sum_{\mu,\rho} f(\mu,-\rho) K_\rho \otimes K_\mu\right).$$

Proof. (a) From [5, Theorem VIII.2.4(b)] we get $(S \otimes \mathrm{id})R = R^{-1}$. Then direct calculation proves (25).

(b) We show that $\tilde{R}_0 = \sum f(\mu,-\rho)K_\mu \otimes K_\rho$ is an inverse of R_0. We have

$$R_0\tilde{R}_0 = \sum_{\substack{\mu_1+\mu_2=\mu\\ \rho_1+\rho_2=\rho}} f(\mu_1,\rho_1) f(\mu_2,-\rho_2) K_\mu \otimes K_\rho$$

and

$$\sum_{\substack{\mu_1+\mu_2=\mu\\ \rho_1+\rho_2=\rho}} f(\mu_1,\rho_1) f(\mu_2,-\rho_2) \underset{(27)}{=} \sum_{\rho_1+\rho_2=\rho} \delta_{\rho_1,-\rho_2} f(\mu,\rho_1) \underset{(28)}{=} \delta_{\rho,0}\delta_{\mu,0}.$$

This implies $R_0\tilde{R}_0 = 1 \otimes 1$. Since R_0 and $\tilde{R}_0$ commute, $\tilde{R}_0$ is the inverse. A similar computation gives the second expression for the inverse. Combining both gives $f(\mu,\rho) = f(-\mu,-\rho)$.

(c) This follows from Remark 8.6. ∎

PROPOSITION 8.14. *Assume that there are no weak diagram antimorphisms of* $(I,\cdot)$ *(e.g., if it is irreducible). Two quasitriangular structures* $R_0\overline{\Theta}$ *and* $R_0'\overline{\Theta}$, *where* $R_0 = \sum f(\mu,\rho)K_\mu \otimes K_\rho$ *and* $R_0' = \sum f'(\mu,\rho)K_\mu \otimes K_\rho$ *are equivalent, i.e., there is a Hopf automorphism* ϕ *such that* $(\phi \otimes \phi)(R_0\overline{\Theta}) = R_0'\overline{\Theta}$ *if and only if there exist isomorphisms* $\gamma_1\colon Y_1 \to Y_1'$, $\gamma_2\colon Y_2 \to Y_2'$ *such that* $f(\mu,\rho) = f'(\gamma_1(\mu),\gamma_2(\rho))$ *and* $\phi(K_\mu) = K_{\gamma_1(\mu)}$, $\phi(K_\rho) = K_{\gamma_2(\rho)}$ *for all* μ,ρ. *The same holds for equivalence of* $\tau(\Theta)\tau(R_0)$ *and* $\tau(\Theta)\tau(R_0')$. *Two quasitriangular structures of the types* $R_0\overline{\Theta}$ *and* $\tau(\Theta)\tau((R_0')^{-1})$ *are never equivalent.*

Proof. Proposition 8.7 implies $(\phi \otimes \phi)\overline{\Theta} = \overline{\Theta}$ and $(\phi \otimes \phi)\tau(\Theta) = \tau(\Theta)$. Hence we need $(\phi \otimes \phi)(R_0) = R_0'$ in both cases which is equivalent to the conditions in the assertion. The last statement holds because $R_0\overline{\Theta} \subset \tilde{u}^-\tilde{u}^0 \otimes \tilde{u}^+\tilde{u}^0$, $\tau(\Theta)\tau((R_0')^{-1}) \in \tilde{u}^+\tilde{u}^0 \otimes \tilde{u}^-\tilde{u}^0$ and each automorphism maps $\tilde{u}^\pm$ to $\tilde{u}^\pm$. ∎

We give explicit examples.

THEOREM 8.15. *Let* $Y_1 = Y_2$ *be subgroups of the group generated by* I *in* $\tilde{Y}$, *which contain the images of* $[i] \in Y$ *for all* $i \in I$ *and satisfy Condition*

(29). *Define a $\mathbb{Z}$-bilinear map $\mathbb{Z}[I]\times\mathbb{Z}[I]\to\mathbb{Q}$ by*

$$i\diamond j := \frac{i\cdot j}{(1/2)(i\cdot i)(1/2)(j\cdot j)} = \frac{\langle i, j'\rangle}{(1/2)j\cdot j} = \frac{\langle j, i'\rangle}{(1/2)i\cdot i}.$$

Let g be the lowest common denominator of the numbers $\{\nu\diamond\tilde{\nu}\mid \nu,\tilde{\nu}\in Y_1\}$. Assume that there exists $\hat{q}\in k$ such that $\hat{q}^g = q$. Define $Z := \{\mu\in Y_1\mid \forall\rho\in Y_1\colon\ \hat{q}^{g\mu\diamond\rho}=1;\ K_\mu$ central$\}$, let $l=|Y_1/Z|$, and assume that char(k) *does not divide l. Then $\tilde{u} := \tilde{\mathrm{u}}/(\{K_\mu - 1\mid \mu\in Z\})$ has a quasitriangular structure as above where $f(\mu,\rho)=\hat{q}^{-g\mu\diamond\rho}/l$.*

Proof. f is well-defined on $Y_1\times Y_1$, because (29) holds. Since Z only contains elements μ such that $\hat{q}^{g\mu\diamond\rho}=1$ for all $\rho\in Y_1$, and $\diamond$ is symmetric, f is well-defined on $Y_1/Z\times Y_1/Z$. Since f is symmetric in its arguments, it suffices to prove the first conditions of (26), (27). For all $i\in I$ we have

$$f(\mu+[j],\rho) = \frac{\hat{q}^{-g\mu\diamond\rho}\hat{q}^{-g[j]\diamond\rho}}{l} = f(\mu,\rho)\hat{q}^{-g\langle\rho,j'\rangle} = f(\mu,\rho)q^{-\langle\rho,j'\rangle}.$$

Moreover

$$\sum_{\rho_1+\rho_2=\rho} f(\mu,\rho_1)f(\tilde{\mu},\rho_2) = \frac{1}{l^2}\sum_{\rho_1+\rho_2=\rho}\hat{q}^{-g\mu\diamond(\rho_1+\rho_2)-g(\tilde{\mu}-\mu)\diamond\rho_2}$$
$$= \frac{1}{l}f(\mu,\rho)\sum_{\rho_2}\hat{q}^{-g(\tilde{\mu}-\mu)\diamond\rho_2}.$$

For all $\mu\in Y_1$, the map $Y_2\to k\setminus\{0\}$, $\rho\mapsto\hat{q}^{-g\mu\diamond\rho}$ is a group character, hence (27) follows from $|Y_1|=|Y_2|$ under the condition that this character is non-trivial for $\mu\notin Z$. Let $\mu\in Y_1$. If K_μ is not central, then there exists $i\in I$ such that $E_iK_\mu\neq K_\mu E_i = q^{\langle\mu,i'\rangle}E_iK_\mu$ (or $F_iK_\mu\neq K_\mu F_i$), whence $\hat{q}^{g\mu\diamond[i]}\neq 1$. If K_μ is central and $\hat{q}^{g\mu\diamond\rho}=1$ for all $\rho\in Y_1$, then $\mu\in Z$ and $K_\mu=1$ in $\tilde{u}$. Condition (28) is automatically satisfied. ∎

EXAMPLE 8.16. (a) The "standard" quasitriangular structure (e.g., [3, Proposition 9.3.9]). If $G\subset\tilde{Y}$ contains all central group-like elements of the form $\tilde{K}_\nu$, Y_1 is generated by the images of $[i]$ in $\tilde{Y}/G$, and char(k) does not divide $|Y_1|$, then $\tilde{u}=\tilde{\mathrm{u}}/(\{K_\mu-1\mid\mu\in G\})$ has quasitriangular structures as above where

$$R_0 = \frac{1}{|Y_1|}\sum_{\nu,\tilde{\nu}\in Y_1} q^{-\nu\cdot\tilde{\nu}}\tilde{K}_\nu\otimes\tilde{K}_{\tilde{\nu}}.$$

(b) Assume that $\frac{i\cdot i}{2}$ is coprime to the order of q for all $i\in I$ and that G contains all central group-like elements of the form K_ν, $\nu\in\mathbb{Z}[I]$. Let Y_1 be the image of $\mathbb{Z}[I]$ in $\tilde{Y}/G$. Assume that char(k) does not divide $|Y_1|$.

Then $\tilde{u}$ has quasitriangular structures as above where

$$R_0 = \frac{1}{|Y_1|} \sum_{\nu, \tilde{\nu} \in Y_1} \hat{q}^{-g\nu \diamond \tilde{\nu}} K_\nu \otimes K_{\tilde{\nu}},$$

where g, $\diamond$ are as in Theorem 8.15 and $\hat{q}$ is a power of q.

Proof. (a) For all $i, j \in I$ we have $\lceil i \rfloor \diamond \lceil j \rfloor = \langle \lceil i \rfloor, j' \rangle = i \cdot j \in \mathbb{Z}$, hence $g = 1$. Moreover, $q^{2l_j \langle \lceil i \rfloor, j' \rangle} = (q_j^{2l_j})^{\langle j, i' \rangle} = 1$. We have $Z = \{0\}$ because $\tilde{u}$ does not contain nontrivial central group-like elements of the form $\tilde{K}_\nu$ where $\lceil \nu \rfloor \in Y_1$.

(b) Since $\frac{i \cdot i}{2}$ is coprime to the order of q, q_i has the same order as q and $\hat{q}$ can be chosen as a power of q. Therefore $q^{2l_i} = 1$, and (29) is satisfied. $\tilde{u}$ does not contain nontrivial central group-like elements of the form K_ν, $\nu \in \mathbb{Z}[I]$, whence $Z = \{0\}$. ∎

Here we give a more explicit description of the quasitriangular structures from the classification in [4]. We recall the situation of the paper in our setting. The element "$\sqrt{q}$" in [4] is called q here. Its order is N as in [4]. We have the Cartan datum $(I, \cdot)$ given by $I = \{i_1, \ldots, i_{n-1}\}$, and

$$i_\alpha \cdot i_\beta = \begin{cases} 4 & \text{if } \alpha = \beta \\ -2 & \text{if } |\alpha - \beta| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we have $Y = \mathbb{Z}[I]$, $K_i^N = 1$, and $q^8 \neq 1$. We will call a quasitriangular structure *minimal* in the sense of [11], that is, if there is no proper Hopf subalgebra A such that the quasitriangular structure is an element of $A \otimes A$. In particular, we have $Y_1 = \tilde{Y}/G$ for minimal quasitriangular structures. Denote the highest common divisor of two positive integers a, b by $\operatorname{hcd}(a, b)$. The classification is given in [4, Theorem 4.1]:

THEOREM 8.17. *Let $Y_0 = 2\tilde{Y}$ be the subgroup generated by $\lceil i \rfloor$ for $i \in I$. Let $\omega = i_1 + i_3 + \cdots + i_{n-1}$ if n is even; let Y_ω be generated by Y_0 and ω and assume that* $\operatorname{char}(k)$ *does not divide $|Y_\omega|$ and $|Y_0|$.*

(a) *If $n > 2$ and* $\operatorname{hcd}(n, N) = 1$, *then* ũ *admits two non-isomorphic quasitriangular structures for $Y_1 = Y_2 = Y_0$ (being minimal if and only if N is odd).*

(b) *If* $\operatorname{hcd}(n, N) = 2$ *and $\frac{N}{2}$ is odd then* ũ *admits two non-isomorphic not minimal quasitriangular structures for $Y_1 = Y_2 = Y_0$ and two non-isomorphic quasitriangular structures for $Y_1 = Y_2 = Y_\omega$ (being minimal if and only if $n = 2$).*

(c) *If* $\operatorname{hcd}(n, N) = 2$ *and* $\frac{N}{2}$ *is even then* $\tilde{\mathfrak{u}}$ *admits four non-isomorphic quasitriangular structures for* $Y_1 = Y_2 = Y_\omega$ (*being minimal if and only if* $n = 2$).

Proof. We only show the existence. For each quasitriangular structure R, we consider the non-isomorphic quasitriangular structure $\tau(R^{-1})$, too (cf. Propositions 8.13 and 8.14).

(a) By [10, 4.3(c)] and $K_i^N = 1$ for all i, $\tilde{\mathfrak{u}}$ does not contain central group-like elements (except 1). If N is odd then it follows from $K_i = \tilde{K}_i^{(N+1)/2}$ that these quasitriangular structures are minimal.

(b) and (c) We show first that the quasitriangular structures for Y_ω are well-defined. We have

$$\omega \diamond \omega = \frac{n}{2} \in \mathbb{N}, \qquad \omega \diamond \lceil i_s \rfloor = \begin{cases} 2 & \text{if } s \text{ is odd,} \\ -2 & \text{if } s \text{ is even} \end{cases}$$

and $q^{2 l_i \lceil i \rfloor \diamond \omega} = 1$ since $q_i = q^2$. Hence the lowest common denominator g is equal to 1 and $f(\mu, \rho) = q^{-\mu \diamond \rho}$ for $\mu, \rho \in Y_1$. Now we show that $Z = \{0\}$, i.e., no central group-likes have to be divided out. Since we have $2\omega \in Y_0$ and Z is a group, it suffices to consider elements of the types $\omega + \lceil \nu \rfloor$, where $\nu \in Y_0$ such that $\tilde{K}_{2\nu} = 1$ and $\lceil \nu \rfloor$ where $\nu \in Y_0$. If $\frac{N}{2}$ is odd then $\tilde{K}_\nu$ has odd order, hence $\omega + \lceil \nu \rfloor = \omega$, but

$$q^{\omega \diamond \lceil i_1 \rfloor} = q^2 \neq 1.$$

If $\frac{N}{2}$ is even then $\frac{n}{2}$ must be odd and

$$q^{(\omega + \lceil \nu \rfloor) \diamond \omega N/2} = (-1)^{n/2} \neq 1$$

because $\lceil \nu \rfloor \diamond \omega$ is even. Now we examine elements of the second type, in particular $\tilde{K}_\nu$ is central. We consider the system of linear equations in [10, 4.3(b)] for central group-like elements and write $N = 2^r t$ where $r \in \mathbb{N}_0$ and t is odd. By the Chinese remainder theorem it is enough to solve the system modulo 2^r and modulo t. But n is coprime to t, whence all entries in the solutions must be multiples of t and the order of a central group-like element is a power of 2. If there are no such elements except 1, we are done, e.g., if $\frac{N}{2}$ is odd, then all elements $\tilde{K}_\nu$ have odd order. Otherwise, $\frac{n}{2}$ is odd and we replace the element by a power of it, which has order 2. It is shown in the proof of Theorem 2.2 in [4] that it must be equal to $K_\omega^{N/2}$. But $q^{\omega N/2 \diamond \omega} = (-1)^{n/2} \neq 1$ since $\frac{n}{2}$ is odd.

Hence we have found all quasitriangular structures if $\frac{N}{2}$ is odd and it is clear that they are minimal if and only if $n = 2$. It remains to show that the quasitriangular structures are not equivalent if $\frac{N}{2}$ is even. By [10, 5.8] for each automorphism ϕ of $\tilde{\mathfrak{u}}$ there exists a diagram automorphism σ of I (either $\sigma = \mathrm{id}$ or $\sigma(i_t) = i_{n-t}$ for all t) such that $\phi(K_i^2) = K_{\sigma(i)}^2$ for all $i \in I$. Moreover $\phi(K_i) K_{\sigma(i)}^{-1}$ is central group-like with square 1, whence $\phi(K_i) \in \{K_{\sigma(i)}, K_{\sigma(i)} K_\omega^{N/2}\}$ and $\phi(K_\omega) \in \{K_\omega, K_\omega^{N/2+1}\}$. In both cases, f remains unchanged because $\frac{N}{2}$ is even and $q^{(N/2+1)\omega \diamond (N/2+1)\omega} = q^{\omega \diamond \omega}$. ∎

8.3. *Ribbon Elements*

We prove that all quasitriangular structures which have been considered in the previous subsection admit ribbon elements. We recall the definition of ribbon elements.

DEFINITION 8.18. Let H be a quasitriangular Hopf algebra with quasitriangular structure $R = \sum R^{(1)} \otimes R^{(2)}$. Define the element $u := \sum S(R^{(2)})R^{(1)}$. The algebra H is called a *ribbon Hopf algebra* if there exists an invertible central element $\mathbf{v} \in H$ (which will be called a *ribbon element*) such that

$$\mathbf{v}^2 = uS(u), \qquad S(\mathbf{v}) = \mathbf{v}, \qquad \varepsilon(\mathbf{v}) = 1, \qquad \Delta(\mathbf{v}) = (\tau(R)R)^{-1}(\mathbf{v} \otimes \mathbf{v}).$$

We recall some properties of R and the element u of Definition 8.18.

LEMMA 8.19. *Let $R = \sum R^{(1)} \otimes R^{(2)}$, u be as in Definition* 8.18. *Then*

$$\begin{aligned} R^{-1} &= (S^{-1} \otimes \mathrm{id})R = (\mathrm{id} \otimes S)R \\ R &= (S \otimes S)R && (30) \\ S^2(x) &= uxu^{-1} \qquad \text{for all } x \in H && (31) \\ \Delta(u) &= (u \otimes u)(\tau(R)R) = (\tau(R)R)(u \otimes u), \qquad \varepsilon(u) = 1 \\ \Delta(S(u)) &= (S(u) \otimes S(u))(\tau(R)R) = (\tau(R)R)(S(u) \otimes S(u)) \\ u^{-1} &= \sum S^{-1}(R^{(1)})S(R^{(2)}). \end{aligned}$$

Proof. See [5, Propositions VIII.4.1 and VIII.4.5]. ∎

LEMMA 8.20. *Let R be a quasitriangular structure and $\check{R} = \tau(R^{-1})$. Compute the elements u and $\check{u}$ from R and $\check{R}$ as in Definition* 8.18. *Then*

(a) $u = S(\check{u})^{-1}$, $S(u) = \check{u}^{-1}$.

(b) *If $\mathbf{v}$ is a ribbon element for R, then $\mathbf{v}^{-1}$ is a ribbon element for $\check{R}$ and vice versa.*

Proof. (a) It suffices to prove $S(u) = \check{u}^{-1}$, because $S^2(u) = u$, $S^2(\check{u}) = \check{u}$, cf. (31). We have

$$\check{u}^{-1} = \sum S^{-2}(S(R^{(2)}))R^{(1)} = \sum S^{-1}(S(R^{(1)})R^{(2)}) = S^{-1}(u) = S(u).$$

(b) The conditions in Definition 8.18 can be checked using $\check{u}S(\check{u}) = S(u)^{-1}u^{-1} = (uS(u))^{-1}$ and $(\tau(\check{R})\check{R})^{-1} = \tau(R)R$. ∎

We recall some facts about root systems using slightly modified notions of [6].

DEFINITION–LEMMA 8.21. *For all $j \in I$ define the maps $(s_j)_Y\colon Y \to Y$ by $(s_j)_Y(\mu) = \mu - \langle \mu, j' \rangle j$ for $\mu \in Y$ and $(s_j)_X\colon X \to X$ by $(s_j)_X(\lambda) = \lambda - \langle j, \lambda \rangle j'$ for $\lambda \in X$. These maps induce actions of the Weyl group W on X and on Y: $(w, \lambda) \mapsto w_X(\lambda), (w, \mu) \mapsto w_Y(\mu)$. For all $\lambda \in X, \mu \in Y$ we have $\langle w_Y(\mu), \lambda \rangle = \langle \mu, w_X^{-1}(\lambda) \rangle$* [6, *Paragraph* 2.2.6]. *Since $(I, \cdot)$ is of finite type, $\mathbb{Z}[I]$ can be canonically imbedded into X and Y. Let W act on $\mathbb{Z}[I]$ via $w_X(i)' = w_X(i')$. Following* [6, *Paragraph* 2.3.1], *we define the sets $\tilde{\mathscr{R}}$ and $\mathscr{R}$ of roots and coroots, respectively, as sets of all $\lambda \in X$ and $\mu \in Y$ where $\lambda = w_X(i')$, $\mu = w_Y(i)$ for some $i \in I$, $w \in W$. Now $\tilde{\mathscr{R}}$ and $\mathscr{R}$ are finite subsets of $\sum_i \mathbb{Z} i' \subset X$ and $\sum_i \mathbb{Z} i \subset Y$. Let $\tilde{\mathscr{R}}^+ = \tilde{\mathscr{R}} \cap \mathbb{N}_0[I]$ and $\mathscr{R}^+ = \mathscr{R} \cap \mathbb{N}_0[I]$ be the sets of positive roots and coroots, respectively. Let $\gamma \in \mathbb{N}_0[I] \subset X$ be the sum of all positive roots. Then*

(a) *$\gamma \cdot i = i \cdot i$ for all $i \in I$.*

(b) *For each root $w_X(i)$ and the corresponding coroot $w_Y(i)$ we have $w_Y(\lceil i \rfloor) = \lceil w_X(i) \rfloor$.*

Proof. (a) This is a property of all root systems.

(b) Direct calculation shows that for all $\nu, \rho \in \mathbb{Z}[I]$ and $l \in I$ we have $(s_l)_X(\nu) \cdot \rho = \nu \cdot (s_l)_X^{-1}(\rho)$. Hence for each $j \in I$ we conclude

$$\begin{aligned}\langle w_Y(\lceil i \rfloor), j' \rangle &= \frac{i \cdot i}{2} \langle i, w_X^{-1}(j)' \rangle = i \cdot w_X^{-1}(j)\\ &= w_X(i) \cdot j = \langle \lceil w_X(i) \rfloor, j' \rangle.\end{aligned}$$

Since $w_Y(\lceil i \rfloor)$ and $\lceil w_X(i) \rfloor$ are in $\mathbb{Z}[I]$ and the Cartan matrix is invertible, they must be equal. ∎

LEMMA 8.22. *Let $\gamma \in \mathbb{Z}[I]$ be as in Definition–Lemma* 8.21. *Then for all $x \in \tilde{\mathfrak{u}}$*

$$S^2(x) = \tilde{K}_\gamma^{-1} x \tilde{K}_\gamma.$$

Proof. It suffices to check it on the algebra generators, where it is an easy calculation. ∎

THEOREM 8.23. *All quasitriangular structures considered in the previous subsection admit ribbon elements.*

Proof. By Lemma 8.20 it suffices to consider quasitriangular structures $R = R_0 \overline{\Theta}$, where $R_0 = \sum f(\mu, \rho) K_\mu \otimes K_\rho$. We write $\overline{\Theta} = \sum \overline{\Theta}^{(1)} \otimes \overline{\Theta}^{(2)}$. We use (30) and obtain

$$u = \sum f(\mu, \rho) K_\rho \overline{\Theta}^{(2)} S^{-1}\left(K_\mu \overline{\Theta}^{(1)}\right) = \sum f(\mu, \rho) K_\rho \overline{\Theta}^{(2)} S^{-1}(\overline{\Theta}^{(1)}) K_{-\mu}.$$

If a summand in $\overline{\Theta}^{(2)}$ is in $\tilde{u}_\nu^+$, then the corresponding summand in $\overline{\Theta}^{(1)}$ and of $S^{-1}(\overline{\Theta}^{(1)})$ belongs to $\tilde{u}_\nu^-\tilde{u}^0$. Hence every group-like element commutes with $\vartheta := \sum \overline{\Theta}^{(2)} S^{-1}(\overline{\Theta}^{(1)})$, and

$$u = \sum f(\mu, \rho) K_{\mu-\rho}\vartheta = \left(\sum f(\mu, \rho) K_{\mu+\rho}\right)^{-1}\vartheta$$

by Proposition 8.13(b); this proposition also implies

$$S\left(\sum f(\mu, \rho) K_{\mu+\rho}\right) = \sum f(\mu, \rho) K_{-\mu-\rho} = \sum f(\mu, \rho) K_{\mu+\rho}.$$

Hence

$$S(u) = \left(\sum S\left(f(\mu, \rho) K_{\mu+\rho}\right)\right)^{-1} S(\vartheta) = \left(\sum f(\mu, \rho) K_{\mu+\rho}\right)^{-1} S(\vartheta).$$

$u^{-1}S(u)$ is group-like by Lemma 8.19. Hence there exists $\mu \in Y$ such that $S(u) = K_\mu u$. We compute K_μ. There exists $\nu \in \mathbb{N}_0[I]$ with maximal trace $\operatorname{tr}\nu$ such that $\tilde{\mathbf{f}}_\nu \neq \{0\}$, because $\tilde{\mathbf{f}}$ is finite-dimensional. Now $S^{-1}(\overline{\Theta}_\nu^{(1)}) = \check{\Theta}_\nu^{(1)}\tilde{K}_\nu$, where $\check{\Theta}_\nu^{(1)} \in \tilde{u}_\nu^-$, whence we have the summand $\tilde{K}_\nu \overline{\Theta}_\nu^{(2)} \check{\Theta}_\nu^{(1)}$ of ϑ. On the other hand, $S(\vartheta) = \sum \overline{\Theta}^{(1)} S(\overline{\Theta}^{(2)})$ and $S(\overline{\Theta}_\nu^{(2)}) = \check{\Theta}_\nu^{(2)}\tilde{K}_{-\nu}$, where $\check{\Theta}_\nu^{(2)} \in \tilde{\mathrm{u}}_\nu^+$, whence we have the summand $\tilde{K}_{-\nu}\overline{\Theta}_\nu^{(1)}\check{\Theta}_\nu^{(2)}$ of $S(\vartheta)$. When we use the commutation formulae for elements of the $+$ and $-$ parts of $\tilde{u}$ (cf. [7, 6.5(a2)] or [6, Corollary 3.1.9]), contributions of homogeneous components of lower trace of degree do not affect the contributions of degree ν. Hence $K_\mu = \tilde{K}_{-\nu}^2$. We choose γ as in Definition–Lemma 8.21. If we can show that $\tilde{K}_{-2\nu-2\gamma}$ is the square of a central group-like element K, then $\mathbf{v} := K\tilde{K}_\gamma u$ is a ribbon element—it is central because for all $x \in \tilde{u}$,

$$S^2(x) = uxu^{-1} = \tilde{K}_\gamma^{-1} x \tilde{K}_\gamma$$

by Lemmas 8.22 and 8.19, and

$$S(\mathbf{v}) = K^{-1}\tilde{K}_\gamma^{-1} S(u) = K^{-1}\tilde{K}_\gamma^{-1}\tilde{K}_{-\nu}^2 u = \mathbf{v};$$

the other conditions follow from Lemma 8.19. It remains to find K. It suffices to consider an irreducible Cartan datum, for if $I = I_1 \dot{\cup} I_2$ where $i_1 \cdot i_2 = 0$ for all $i_1 \in I_1, i_2 \in I_2$, and if $\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2$ are the corresponding algebras and K_1 and K_2 central group-like elements (which are central in $\tilde{u}$, too) with the desired properties for $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$, then it follows from $\tilde{\mathbf{f}} \cong \tilde{\mathbf{f}}_1 \otimes \tilde{\mathbf{f}}_2$ (cf. proof of [10, 2.14]) that $K = K_1 K_2$ is a central group-like element which satisfies the desired properties for $\tilde{\mathbf{f}}$.

(a) Assume that q fits to ${}_k\mathbf{f}$. We have a Poincare–Birkhoff–Witt type basis of $\tilde{\mathbf{f}}$ (cf. [10, Subsections 1.4 and 2.9]) consisting of elements $b_{\mathbf{c}}$, where $c_{i_l} < l_{i_l}$ for all l, where l_{i_l} is the order of $q_{i_l}^2$. The degree of

$$T'_{i_1,e} T'_{i_2,e} \cdots T'_{i_{l-1},e}\left(E_{i_l}^{(c_l)}\right)$$

for $c_l = 1$ is $\alpha_l := (s_{i_1})_X \cdots (s_{i_{l-1}})_X(i_l)$, where α_l runs through all positive roots in $\tilde{\mathscr{R}}^+$. Now $c_l < l_{i_l} =: l_{\alpha_l}$ (this is independent of the actual choice of reduced expression of the positive coroot α_l). Hence the maximal degree is

$$\nu = \sum_{\alpha \in \tilde{\mathscr{R}}^+} \alpha(l_\alpha - 1).$$

Therefore $2\gamma + 2\nu = \sum_{\alpha \in \tilde{\mathscr{R}}^+} 2l_\alpha \alpha$, and $\tilde{K}_{-2\nu-2\gamma} = 1$ because for each positive root $\alpha = w_X(i)$ we have $\tilde{K}_\alpha^{2l_\alpha} = K_{w_Y(i)}^{i\cdot il_i}$ by Definition–Lemma 8.21(b) and because for $w_Y(i) = (s_{i_1})_Y \cdots (s_{i_{l-1}})_Y(i_l)$ we have

$$K_{w_Y(i)}^{i\cdot il_i} = T'_{i_1,e} \cdots T'_{i_{l-1},e}\left(K_i^{i\cdot il_i}\right) = 1.$$

Therefore $K = 1$ is an appropriate choice.

(b) If q does not fit to ${}_k\mathbf{f}$, then [10, 2.13] applies. Here $\nu = 2i + 2j$ and $\tilde{K}_{-\nu-\gamma}$ is central because

$$q^{-(\nu+\gamma)\cdot i} = q^{-2i\cdot i} = 1, \qquad q^{-(\nu+\gamma)\cdot j} = q^{-2j\cdot j} = 1.$$

Therefore K can be chosen as $\tilde{K}_{-\nu-\gamma}$. ∎

COROLLARY 8.24. *The Frobenius–Lusztig kernel for sl_n considered in* [4] *and Theorem* 8.17 *and the quotients considered in Example* 8.16 *are ribbon Hopf algebras for all quasitriangular structures of the type described in Theorem* 8.11 *and Proposition* 8.13.

8.4. *Quasitriangular Structures for ${}_k\mathbf{U}$*

We show that ${}_k\mathbf{U}$ becomes a topological quasitriangular Hopf algebra (cf. [3, remarks after 9.3.9]). We complete the tensor products ${}_{\mathscr{A}}\mathbf{U} \otimes_{\mathscr{A}} \mathbf{U}$ and ${}_k\mathbf{U} \otimes_k \mathbf{U}$ in an analogous way as described in [6, 4.1.1] and denote them by $({}_{\mathscr{A}}\mathbf{U} \otimes_{\mathscr{A}} \mathbf{U})^\wedge$ and $({}_k\mathbf{U} \otimes_k \mathbf{U})^\wedge$, respectively. In [10, 2.9] elements $b_{\mathbf{c}}, \tilde{b}_{\mathbf{c}} \in \mathbf{f}$ for $\mathbf{c} \in \mathbb{N}_0^n$ (where n depends on the Cartan datum) have been considered where $(b_{\mathbf{c}}, \tilde{b}_{\tilde{\mathbf{c}}}) = \delta_{\mathbf{c},\tilde{\mathbf{c}}}$ for all $\mathbf{c}, \tilde{\mathbf{c}} \in \mathbb{N}_0^n$. Therefore for $\nu \in \mathbb{N}_0[I]$ the sums

$$\Theta_\nu = (-1)^{\operatorname{tr}\nu} v_\nu \sum_{\mathbf{c}:\, b_{\mathbf{c}} \in \mathbf{f}_\nu} b_{\mathbf{c}}^- \otimes \tilde{b}_{\mathbf{c}}^+$$

(recall that $v_\nu = \prod_i v_i^{\nu_i} = v^{\operatorname{tr}[\nu]}$ for $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$) are summands of a Quasi-$\mathscr{R}$-matrix $\Theta = \sum_\nu \Theta_\nu$ of the completion $({}_{\mathscr{A}}\mathbf{U} \otimes_{\mathscr{A}} \mathbf{U})^\wedge$ as in [6, 4.1.2]. In particular, the properties shown in [6, 4.1.2–3] and analogues of Proposition 8.5 are valid. After tensoring with k over $\mathscr{A}$, we get a Quasi-$\mathscr{R}$-Matrix of ${}_k\mathbf{U}$ which will be denoted by Θ again and satisfies analogous properties. The following theorem can be proved in a similar way as Theorem 8.11 (note that ${}_k\mathbf{U}$ is generated as algebra by $E_i^{(t)}, F_i^{(t)}, K_\mu$ for $i \in I, t \in \mathbb{N}_0, \mu \in Y$).

THEOREM 8.25. *Let* Y_1, Y_2 *be as in Theorem* 8.11. *Let* $R_0 = \sum_{\mu,\rho} f(\mu, \rho) K_\mu \otimes K_\rho$ *where* $f: Y_1 \otimes Y_2 \to k$ *is any map. The product* $R_0\overline{\Theta}$ *is a quasitriangular structure of* ${}_k\mathbf{U}$ *if and only if* f *satisfies the conditions* (26)–(28).

Note added in proof. If the Carban datum is not of finite type than $\hat{\mathbf{f}}$ and $\hat{\mathfrak{u}}$ are always infinite dimensional.

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